

# Signal analysis through analog representation

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We present an approach to the analysis of signals based on analog representation of measurements. Methodologically, it relies on the consideration and full use of the continuous nature of a realistic, as opposed to an idealized, measuring process. Mathematically, it is based on the transformation of discrete or continuous signals into normalized continuous scalar fields with the mathematical properties of distribution functions. This approach allows a simple and efficient implementation of many traditionally digital analysis tools, including nonlinear filtering techniques based on order statistics. It also enables the introduction of a large variety of new characteristics of both one- and multi-dimensional signals, which have no digital counterparts.

**Keywords:** analog signal processing; nonlinear filters; threshold densities; amplitude density; counting density; analog rank filters

## 1. Introduction

Our world is full of signals, both natural and man made. Since these signals are normally measured as continuously varying quantities, it is not surprising that, in the early days of signal analysis, most devices for processing signals were made with analog, or continuous-action, components. Such devices use basic physical principles and properties of the materials from which they are built, in order to accomplish various signal-processing tasks.

Rapid development of digital technology since the 1950s has changed this situation dramatically. The proliferation of digital methods has become so great that analog devices have fallen out of fashion. Nevertheless, while the conversion to digital technology is undoubtedly justified by the flexibility, universality and low cost of modern integrated circuits, it usually comes at the price of high complexity of both hardware and software implementations. In addition, all digital operations require external power input, while many operations in analog devices can be performed by passive components. Thus, analog devices consume much less energy and are therefore more suitable to operate in autonomous conditions, such as mobile communications, space missions, prosthetic devices, etc. The added complexity of digital devices stems from the fact that all operations must be reduced to the elemental manipulation of binary quantities using primitive logic gates. Therefore, even such basic operations as integration and differentiation of functions require a very large number of such gates

and/or sequential processing of discrete numbers representing the function sampled at many points. On the other hand, the same operations can be performed instantly in an analog device by passing the signal representing the function through a simple RC circuit.

Of course, there are many signal-processing tasks for which digital algorithms are well known, but corresponding analog operations are hard to reproduce. One example, which is widely recognized to fall within this category, is related to the use of signal-processing techniques based on *order statistics*<sup>†</sup>, e.g. implementing median and other order-statistic filtering (Tukey 1977). Order-statistic filters are gaining wider recognition for their ability to provide more robust estimators of signal properties. For example, the *median* value of a set of measurements usually represents the general trend in a signal better than the *mean* value, since the latter is more sensitive to outliers. However, while analog implementation of the mean is trivial, median estimators are much harder to implement in analog form (Jarske & Vainio 1993), since, traditionally, their determination has involved the operation of sorting or ordering a set of measurements. Indeed, there is no conceptual difficulty in sorting a set of discrete measurements, but it is much less obvious how to perform similar operations for continuous signals (Bottema 1991; Ferreira 2000, 2001).

Nevertheless, fuelled by the need for robust filters that can operate in real time and on a low energy budget, analog implementation of traditionally digital operations has recently gained in popularity aided by the rapid progress in analog very large scale integration (VLSI) technology (Kinet & Steyaert 1997; Lee & Jen 1993; Mead 1989; Murthy & Swamy 1992). However, current efforts to implement digital signal-processing methods in analog devices still employ an essentially digital philosophy. That is, a continuous signal is typically passed through a delay line which samples the signal at discrete time intervals. The individual samples are then processed by a cascade of analog devices that mimic elemental digital operations (Li & Holmes 1988; Liu *et al.* 1993). Such an approach fails to exploit the main strength of the analog processing, which is the ability to perform complex operations in a single step, without employing the ‘divide and conquer’ paradigm of the digital approach.

As pointed out by some authors (Paul & Hüper 1993, for example), the major problem in analog rank processing is the lack of an appropriate differential equation for ‘analog sorting’. There have been several attempts to implement such sorting and to build continuous-time rank filters without using delay lines and/or clock circuits. Examples of these efforts include optical rank filters (Ochoa *et al.* 1987), analog sorting networks (Opris 1996; Paul & Hüper 1993) and analog rank selectors based on minimization of a nonlinear objective function (Urahama & Nagao 1995). However, the term ‘analog’ is often perceived as only ‘continuous time’, and thus these efforts fall short of considering the threshold continuity, which is necessary for a truly analog representation of differential sorting operators. Even though Ferreira (2000, 2001) extensively discusses threshold distributions, these distributions are only piecewise continuous and thus do not allow straightforward introduction of differential operations with respect to threshold.

Here we introduce an approach to the analysis of signals which is based on the consideration and full use of the continuous nature of signals registered with real

<sup>†</sup> See, for example, Arnold *et al.* (1992) and Sarhan & Greenberg (1962) for the definitions and theory of order statistics.

(as opposed to idealized) acquisition systems. We address the problem of measurement and analysis of signals on a consistent general basis by introducing *continuous threshold distributions* of various signal properties. As described in more detail in § 3, the threshold distributions result from the *averaging* of *instantaneous* distributions with respect to time, space (in the case of multi-dimensional signals) and thresholds. Since the averaging is performed by a *continuous* kernel (test function), we refer to it as the *analog representation* of measured signals. Such a representation can be viewed as the transformation of discrete or continuous signals into normalized continuous scalar fields, i.e. into objects with the mathematical properties of distribution functions. This approach allows us to successfully overcome the limitations of digital analysis, opening up, among other possibilities, the opportunity to expand order-statistic analysis of signals, and to provide a means for its efficient implementation in both hardware and software.

The rest of this article is organized as follows. In § 2 we consider a simple measurement process involving *ideal* discriminators, and introduce the threshold distribution and density functions of the amplitude of a continuous signal. These functions allow us to define many useful properties of the signal. As an illustrative example, we consider a quantile filter of a continuous signal. Section 3 describes a more realistic measurement of the threshold distributions and densities by means of *real* discriminators, which can be interpreted as threshold averaging of the ideal distributions and densities with a continuous test function. In § 4, we describe several considerations leading towards a practical implementation of analog quantile filters. Section 5 expands the definition of threshold distributions and densities given in §§ 2 and 3 to include distributions of threshold crossings, local extrema and other properties of a signal. This section also describes analog quantile filters for these properties. Section 6 addresses applicability of the analysis in terms of the threshold density functions for spatially dependent signals such as scalar and vector fields. In § 7, we summarize the main principles of the proposed approach and provide additional examples of its usefulness for the analysis of signals by analog means.

Throughout this article, we try to emphasize ideas rather than the details of technical development. The examples we have chosen to illustrate these ideas were prepared by numerically solving the appropriate differential equations.† Specific implementations, algorithms and designs of various components of analog devices based on the proposed approach will be described in future publications.

## 2. Threshold distributions and analog quantile filters

Even though the analysis of a signal is often considered separately from its measurement, a complete understanding of signal properties requires detailed knowledge of the acquisition system which measures the signal. Thus signal measurement and analysis go hand in hand. In a technical sense, measurement is the process of assigning numbers or other symbols as values of a variable in order to establish its relation to a standard unit or to another variable of the same nature. One of the most basic ways to establish such a relation is to compare the signal  $x(t)$  with a *threshold*, or *displacement*, variable  $D$ . In practice, this can be accomplished by means of *discriminators*. An ideal discriminator outputs a ‘0’ or ‘1’, respectively, depending on

† In all presented examples, the precision of the solutions exceeds the graphical resolution of the respective figures.

whether the signal value is larger or smaller than  $D$ . Mathematically, the output of such an ideal discriminator can be conveniently represented by the Heaviside unit step function  $\theta[D - x(t)]$ . Note that such a representation is equivalent to the signal representation as a function of time,  $x(t)$ , the only difference being the introduction of the displacement variable  $D$ . This is similar to the description of algebraic equations by the coordinate method in analytic geometry.†

(a) *Time-dependent threshold distribution*

Despite the apparent simplicity of representing the measurement process by means of ideal discriminators, this method allows us to define many useful signal properties. For instance, the fraction of time in the interval from 0 to  $T$  that a signal spends *below* a given threshold,  $D$ , can be calculated simply as the time average of the step function (see, for example, Nikitin *et al.* 1998),

$$\Phi(D) = \frac{1}{T} \int_0^T dt \theta[D - x(t)]. \quad (2.1)$$

It is important to note that the same expression also defines the *distribution function* of a continuous signal (see, for example, Ferreira 2000, 2001) and, therefore, immediately opens up possibilities to explore the order-statistic properties of a signal. For example, the *median* value of the signal  $x(t)$  in the interval  $[0, T]$  is given by the threshold value  $D = D_m$ , such that

$$\Phi(D_m) = \frac{1}{2}. \quad (2.2)$$

Equation (2.1) for the distribution function can be easily generalized for signals within an arbitrary time window  $w$ , thereby defining the *time-dependent* distribution function as the convolution integral

$$\Phi(D, t) = \int_{-\infty}^{\infty} ds w(t-s) \theta[D - x(s)] = w(t) * \theta[D - x(t)], \quad (2.3)$$

where the time window function is such that  $w(t) \geq 0$  and  $\int_{-\infty}^{\infty} ds w(s) = 1$ , and the asterisk denotes convolution. Note that equation (2.3) represents a measurement of the output of the ideal discriminator  $\theta[D - x(t)]$  with an instrument having the impulse time response  $w$ . The median of  $x(t)$  within the moving window  $w$  is now a function of time,  $D_m = D_m(t)$ , which is defined implicitly by

$$\Phi(D_m, t) = \frac{1}{2}. \quad (2.4)$$

This can be interpreted as the output of the *median filter*. More generally, the *quantile* filter of order  $q$  is given by the function  $D_q(t)$ , defined implicitly as

$$\Phi[D_q(t), t] = q, \quad 0 < q < 1. \quad (2.5)$$

Since the expressions (2.1) and (2.3) for the distribution function are written as time averages of the function  $\theta[D - x(t)]$ , the latter can be interpreted as the *instantaneous* threshold distribution of the signal  $x(t)$ . As noted above, this function is

† It is convenient for our purpose to adopt the definition of the Heaviside step function such that  $\theta(0) \equiv \frac{1}{2}$  (see, for example Bracewell 1978, p. 57). Then the equation  $\theta[D - x(t)] = \frac{1}{2}$  describes  $x(t)$  as a curve in the plane  $(t, D)$ .

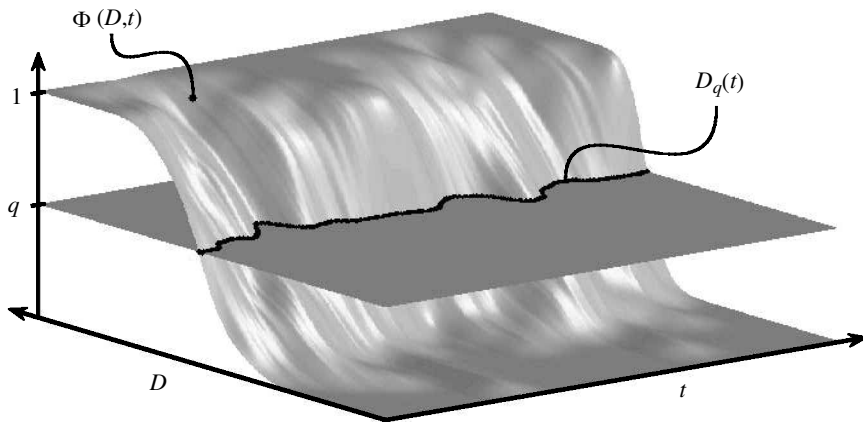


Figure 1. Defining output of a quantile filter  $D_q(t)$  as a level curve of the distribution function  $\Phi(D, t)$ .

simply an alternative representation of the signal. However, since  $\theta[D - x(t)]$  is a function of *two* variables,  $t$  and  $D$ , this representation allows us to formally introduce an additional independent variable—the threshold coordinate  $D$ —in the description of the signal. This introduction of the threshold coordinate provides an additional dimension to the signal, and it is an essential feature of our approach. It enables a simple geometric interpretation of various complicated signal transformations, thereby facilitating further developments of signal-processing techniques.

(b) *Ideal analog quantile filter*

As an illustration, let us consider the quantile filter defined by equation (2.5). Viewing the function  $\Phi(D, t)$  as a surface in the three-dimensional space  $(t, D, \Phi)$ , we immediately have a geometric interpretation of  $D_q(t)$  as that of a level (or contour) curve obtained from the intersection of the surface  $\Phi = \Phi(D, t)$  with the plane  $\Phi = q$ , as shown in figure 1. As is well known from elementary differential geometry (see, for example, Bronshtein & Semendiaev 1997, p. 551, eqn (4.29)), it is possible to obtain an explicit (albeit differential) equation of a level curve by differentiating equation (2.5) with respect to time; specifically

$$\frac{d\Phi}{dt} = \frac{\partial\Phi}{\partial D_q} \frac{dD_q}{dt} + \frac{\partial\Phi}{\partial t} = 0, \quad (2.6)$$

which yields the following differential equation for  $D_q(t)$ :

$$\frac{dD_q}{dt} = -\frac{\partial\Phi/\partial t}{\partial\Phi/\partial D_q}. \quad (2.7)$$

The solution of this equation will follow the level curve corresponding to the output of the quantile filter if we choose the initial condition  $D_q(t_0)$  in which  $\Phi[D_q(t_0), t_0] = q$ . Since it is well known that differential equations are readily reproduced by analog devices (see, for example, McGillem & Cooper 1991), the above representation of the quantile filter in terms of a differential equation offers evidence of the usefulness of our approach.

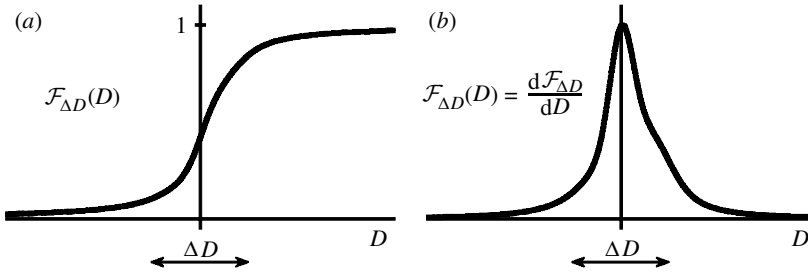


Figure 2. Representative responses of a real (continuous) discriminator and its associated probe: (a) discriminator  $\mathcal{F}_{\Delta D}(D)$ ; (b) probe  $f_{\Delta D}(D) = d\mathcal{F}_{\Delta D}(D)/dD$ .

The numerator and denominator on the right-hand side of equation (2.7) have the following interpretation. The partial derivative of the distribution function  $\Phi(D, t)$  with respect to time is given by

$$\frac{\partial \Phi}{\partial t} = \frac{dw(t)}{dt} * \theta[D - x(t)]. \quad (2.8)$$

It is the time average of the instantaneous threshold distribution in a moving (differentiating) window  $dw/dt$ . The partial derivative of  $\Phi(D, t)$  with respect to threshold,

$$\frac{\partial \Phi}{\partial D} = w(t) * \frac{d}{dD} \theta[D - x(t)], \quad (2.9)$$

can be interpreted as the time-dependent threshold *density* of the signal  $x(t)$  in the moving window  $w$ . This interpretation follows directly from the relationship between the density  $\phi(x)$  and the associated (cumulative) distribution  $\Phi(x)$ , namely  $\phi(x) = d\Phi(x)/dx$ . The derivative of the Heaviside unit step function  $\theta(x)$  is known as the *Dirac  $\delta$ -function*  $\delta(x)$ ,<sup>†</sup>

$$\frac{d\theta(x)}{dx} = \delta(x). \quad (2.10)$$

The expression for the threshold density of the signal  $x(t)$  in a moving window  $w$  can thus be written as

$$\phi(D, t) = \int_{-\infty}^{\infty} ds w(t-s) \delta[D - x(s)] = w(t) * \delta[D - x(t)], \quad (2.11)$$

which is the time average of the *instantaneous* threshold density  $\delta[D - x(t)]$ . Note that, even though the properties of the threshold distribution and density defined above are usually associated with those of the probability distribution and density, the above definitions are given for deterministic signals and do not rely on the usual axioms of probability and statistics.

### 3. Real discriminators and probes

The insightful reader will immediately raise an objection that equation (2.7) is hardly suitable for determining an output of a quantile filter in practice. Indeed, as should

<sup>†</sup> See, for example, Dirac (1958) or Arfken (1985) for the definition and properties of the Dirac  $\delta$ -function. Also note that since the Dirac  $\delta$ -function is an *even* function equation (2.10) implies that  $\theta(0) \equiv \frac{1}{2}$ .

be clear from (2.11), the integrand in the denominator of the right-hand side of this equation cannot be evaluated directly, since it contains a singular Dirac  $\delta$ -function.<sup>†</sup> It is important to realize, however, that this difficulty is not inherent in our approach. Rather, it is the result of the approximation we adopted when considering the measurement process that employs an *ideal* discriminator, capable of comparison with infinite precision. In reality, measurements with infinite precision are unavailable. All physical observations are limited to a finite resolving power, and the only measurable quantities are the weighted means over non-zero intervals. Therefore, a more realistic representation of the discriminator would be in terms of a *continuous* function,  $\mathcal{F}_{\Delta D}(D)$ . This function changes monotonically from 0 to 1 so that most of this change occurs over some characteristic range of threshold values  $\Delta D$ , as illustrated in figure 2a. The distribution function of the signal  $x(t)$  in a time window  $w$  measured by a real acquisition system is therefore expressed as

$$\Phi(D, t) = w(t) * \mathcal{F}_{\Delta D}[D - x(t)], \quad (3.1)$$

where the exact shape of the real discriminator function  $\mathcal{F}_{\Delta D}(D)$  depends on the properties of the acquisition system. The differential equation for the quantile filter is still given by equation (2.7), where the distribution function  $\Phi(D, t)$  is defined by equation (3.1). As before, the threshold density of a signal is given by

$$\phi(D, t) = \frac{\partial \Phi(D, t)}{\partial D} = w(t) * f_{\Delta D}[D - x(t)], \quad (3.2)$$

where

$$f_{\Delta D}[D - x(t)] = \frac{d}{dD} \mathcal{F}_{\Delta D}[D - x(t)] \quad (3.3)$$

is the instantaneous threshold density of  $x(t)$ , measured by the real acquisition system. Again, the exact shape of the function  $f_{\Delta D}(D)$  will depend on the properties of the acquisition system (that is, on the shape of the discriminator function  $\mathcal{F}_{\Delta D}(D)$ ), but will typically have a pronounced maximum around  $D = 0$  and decay to zero as  $|D| \rightarrow \infty$ . As it follows from (3.3), an appropriate name for a device described by  $f_{\Delta D}(D)$  would be ‘differential discriminator’. However, it is convenient to use a simpler designation ‘probe’, instead of ‘differential discriminator’, for this device. Figure 2b shows the characteristic of the associated probe of the discriminator shown in figure 2a.

Note that, while the discriminator function  $\mathcal{F}_{\Delta D}(D)$  represents the threshold *step* response of the acquisition system, the function  $f_{\Delta D}(D)$  is the system’s threshold *impulse* response. Thus the instantaneous threshold density given by equation (3.3) can be interpreted as the *threshold average*, with respect to the test function  $f_{\Delta D}(D)$ , of the ideal instantaneous density; that is,

$$f_{\Delta D}[D - x(t)] = \int_{-\infty}^{\infty} dr f_{\Delta D}(D - r) \delta[r - x(t)]. \quad (3.4)$$

Most importantly, the threshold-averaged instantaneous density no longer possesses singularities, and thus its evaluation does not present any conceptual difficulty. Therefore, when considered within the framework of a realistic measurement process, the differential equation (2.7) can be used directly for a practical implementation of an analog quantile filter.

<sup>†</sup> It can also be shown that every extremum of a signal within the moving window produces a singularity in the density function given by equation (2.11).

(a) *Explicit expression for an analog quantile filter*

Note, however, that a differential equation is not the only possible embodiment of an analog quantile filter. Other means of locating the level lines of the threshold distribution function can be developed, based on the geometric interpretation discussed above. For example, one can start by using the sifting property of the Dirac  $\delta$ -function to write  $D_q(t)$  as

$$D_q(t) = \int_{-\infty}^{\infty} dD D \delta[D - D_q(t)] \quad (3.5)$$

for all  $t$ . Then, recalling that  $D_q(t)$  is a root of the function  $\Phi(D, t) - q$  and that, by construction, there is only one such root for any given time  $t$ , we can replace the  $\delta$ -function of thresholds with that of the distribution function values as

$$D_q(t) = \int_{-\infty}^{\infty} dD D \phi(D, t) \delta[\Phi(D, t) - q]. \quad (3.6)$$

Here we have used the Dirac  $\delta$ -function property (see, for example, Davydov 1988, p. 610, eqn (A 15))

$$\delta[a - f(x)] = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}, \quad (3.7)$$

where  $|f'(x_i)|$  is the absolute value of the derivative of  $f(x)$  at  $x_i$ , and the sum goes over all  $x_i$  such that  $f(x_i) = a$ . We have also used the fact that  $\phi(D, t) \geq 0$ .

The final step in deriving a practically useful realization of the quantile filter is to replace the  $\delta$ -function of the ideal measurement process with a finite-width pulse function  $g_{\Delta q}$  of the real measurement process, namely

$$D_q(t) = \int_{-\infty}^{\infty} dD D \phi(D, t) g_{\Delta q}[\Phi(D, t) - q], \quad (3.8)$$

where  $\Delta q$  is the characteristic width of the pulse. That is, we replace the  $\delta$ -function with a continuous function of finite width and height. This replacement is justified by the observation made earlier: it is impossible to construct a physical device with an impulse response expressed by the  $\delta$ -function, and thus an adequate description of any real measurement must use the actual response function of the acquisition system instead of the  $\delta$ -function approximation.

It may be argued that the introduction of real discriminators constitutes an approximation to the exact determination of the median and other order-statistic properties of the signal. On the other hand, from a practical point of view, the definitions of various signal properties involve idealized (and, therefore, approximate) models of real measurement systems. The authors adhere to the practical philosophy and believe that signal properties cannot be divorced from the measurement process.

(b) *Analog  $L$  filters and  $\alpha$ -trimmed mean filters*

It is worth pointing out the generalization of analog quantile filters which follows from equation (3.6). In the context of digital filters, this generalization corresponds to the  $L$  filters described in Bovik *et al.* (1983).



Indeed, we can write a linear combination of the outputs of various quantile filters as

$$\begin{aligned} D_L(t) &= \int_0^1 dq W_L(q) D_q(t) = \int_0^1 dq W_L(q) \int_{-\infty}^{\infty} dD D \phi(D, t) \delta[\Phi(D, t) - q] \\ &= \int_{-\infty}^{\infty} dD D \phi(D, t) W_L[\Phi(D, t)], \end{aligned} \quad (3.9)$$

where  $W_L$  is some (normalized) weighting function. Note that the difference between equations (3.9) and (3.8) is in replacing the narrow pulse function  $g_{\Delta q}$  in (3.8) by an *arbitrary* weighting function  $W_L$ .

A particular choice of  $W_L$  in (3.9) as the rectangular (boxcar) probe of width  $1 - 2\alpha$ , centred at  $\frac{1}{2}$ , will correspond to the digital  $\alpha$ -trimmed mean filters

$$\bar{D}_\alpha(t) = \int_{-\infty}^{\infty} dD D \phi(D, t) b_\alpha[\Phi(D, t)], \quad 0 \leq \alpha < \frac{1}{2}, \quad (3.10)$$

described in Bednar & Watt (1984), where

$$b_\alpha(x) = \frac{1}{1 - 2\alpha} [\theta(x - \alpha) - \theta(x - 1 + \alpha)].$$

When  $\alpha = 0$ , equation (3.10) describes the running mean filter,  $\bar{D}_{\alpha=0}(t) = \bar{x}(t)$ , and in the limit  $\alpha \rightarrow \frac{1}{2}$  it describes the median filter,  $\lim_{\alpha \rightarrow 1/2} \bar{D}_\alpha(t) = D_m(t)$ .

#### 4. Practical considerations

Note that a quantile filter expressed by the differential equation (2.7) employs the quantile order  $q$  only via the initial conditions. This is not suitable for practical use, since any deviation from the particular choice of the initial condition will result in a different order quantile filter. Moreover, the presence of noise will inevitably cause the output of the filter to drift away from the chosen value of  $q$ . It is much more desirable to have a filter that converges to the chosen quantile order regardless of the initial condition. In other words, the solution of the differential equation for the chosen quantile must be stable with respect to other quantile values.

There are many ways to achieve such stability, and a particular choice will be governed by practical considerations. One of the simplest possible approaches is to add a term proportional to  $q - \Phi(D_q, t)$  to the right-hand side of equation (2.7), namely

$$\frac{dD_q}{dt} = - \frac{\partial \Phi(D_q, t) / \partial t}{\phi(D_q, t)} + \nu [q - \Phi(D_q, t)], \quad \nu > 0. \quad (4.1)$$

Since  $\Phi(D, t)$  is a monotonically increasing function of  $D$  for all  $t$ , the added term will ensure the convergence of the solution to the chosen quantile order  $q$  regardless of the initial condition. Parameter  $\nu$  in (4.1) is the characteristic convergence speed, in units of 'threshold per time'.

An important practical consideration for the design of any device is the simplicity of its components. From this point of view, equation (4.1) might not be considered to be particularly simple, mainly because of the necessity to evaluate the partial derivatives of  $\Phi$ , as well as the uncertainty about the choice of  $\nu$ . Fortunately, as we

show below, the required design simplification again comes from the consideration of a realistic measurement process. Indeed, the reality of physical measurements is such that any sensor—as well as the acquisition system as a whole—has a certain inertia. The inertial properties of a measuring device are usually described by its transient characteristic, i.e. by the response of the device to a *unit step* of the input signal. Such a response for many physical sensors is well represented by the function  $H_\tau = \theta(t)(1 - e^{-t/\tau})$ , where  $\tau$  is the characteristic response time. This means that the total *impulse* time response of a typical measuring device is given by

$$w(t) = h_\tau(t) * w_T(t), \quad (4.2)$$

where

$$h_\tau = \frac{dH_\tau}{dt} = \theta(t) \exp\left(-\frac{t}{\tau} - \ln \tau\right),$$

and  $w_T$  is the desired (or designed) impulse response of the device. For example, if  $w_T$  is the impulse time response of an electric amplifier, and the resistance  $R$  and capacitance  $C$  of the connecting cable are such that  $RC = \tau$ , then the total impulse time response  $w(t)$  of the apparatus composed of the amplifier and the cable will be given by equation (4.2).

When the time impulse response of the measuring system is given by (4.2), we can relate the derivative of  $w(t)$  to the function itself via

$$\frac{dw}{dt} = \frac{dh_\tau}{dt} * w_T(t) = \frac{1}{\tau}[\delta(t) - h_\tau(t)] * w_T(t) = \frac{1}{\tau}[w_T(t) - w(t)]. \quad (4.3)$$

We can use this formula to rewrite the numerator in equation (2.7) as

$$\frac{\partial \Phi}{\partial t} = \frac{dw(t)}{dt} * \mathcal{F}_{\Delta D}[D - x(t)] = \frac{1}{\tau}\{w_T(t) * \mathcal{F}_{\Delta D}[D - x(t)] - \Phi(D, t)\}. \quad (4.4)$$

This leads to the following differential equation for the analog quantile filter:

$$\frac{dD_q}{dt} = \frac{\Phi(D_q, t) - w_T(t) * \mathcal{F}_{\Delta D}[D_q - x(t)]}{\tau h_\tau(t) * w_T(t) * f_{\Delta D}[D_q - x(t)]}. \quad (4.5)$$

In order to ensure the convergence of the solution to a given quantile order  $q$ , we can replace  $\Phi(D_q, t)$  with  $q$ , arriving at a simple result,

$$\frac{dD_q}{dt} = \frac{q - w_T(t) * \mathcal{F}_{\Delta D}[D_q - x(t)]}{\tau h_\tau(t) * w_T(t) * f_{\Delta D}[D_q - x(t)]}, \quad (4.6)$$

which is equivalent to choosing the characteristic speed of convergence  $\nu$  in equation (4.1) as  $\nu = \{\tau h_\tau(t) * w_T(t) * f_{\Delta D}[D_q - x(t)]\}^{-1}$ .

The shortcoming of a filter given by equation (4.6) is that the convolution integrals on its right-hand side need to be re-evaluated (updated) for each new value of  $D_q$ . Since we would rather employ a filter in a simple feedback circuit, the final step in the practical implementation of an analog quantile filter should be to replace the right-hand side of equation (4.6) by an approximation which can be easily evaluated by such a circuit. Of course, one can employ a great variety of such approximations (see, for example, Bleistein & Handelsman 1986; Copson 1967; Erdélyi 1956), whose suitability will depend on the particular goal. Although a detailed discussion of these approximations is beyond the scope of this article, we provide an example of one such approximation in the next section.

## (a) Analog rank selectors

As an illustration, let us consider the impulse time response function  $w$  composed of a (finite) train of inertial impulse response functions  $h_\tau$  as

$$w(t) = \sum_k w_k h_\tau(t - t_k) = h_\tau(t) * \sum_k w_k \delta(t - t_k), \quad (4.7)$$

where  $\sum_k w_k = 1$ . In other words, the desired impulse time response of the device is  $w_T = \sum_k w_k \delta(t - t_k)$ .

Now notice that when  $\tau$  is sufficiently small (i.e. when  $x(t)$  is well approximated, about any  $t$ , by the first few terms of its expansion in powers of  $\tau^\dagger$ ), the only significant contribution to the integral  $h_\tau(t) * f_{\Delta D}[D_q - x(t)]$  will come from the immediate vicinity of the point  $s = t$ . Thus we can replace this integral by its approximation as

$$\begin{aligned} h_\tau(t) * f_{\Delta D}[D_q - x(t)] &= \frac{1}{\tau} \int_{-\infty}^t ds \exp\left(\frac{s-t}{\tau}\right) f_{\Delta D}[D_q(t) - x(s)] \\ &\approx \frac{1}{\tau} \int_{-\infty}^t ds \exp\left(\frac{s-t}{\tau}\right) f_{\Delta D}[D_q(s) - x(s)] \\ &= h_\tau(t) * f_{\Delta D}[D_q(t) - x(t)]. \end{aligned} \quad (4.8)$$

Substitution of (4.7) and (4.8) into equation (4.6) leads to the approximate expressions for a quantile filter,

$$\frac{dD_q}{dt} \approx \frac{q - \sum_k w_k \mathcal{F}_{\Delta D}[D_q(t) - x(t - t_k)]}{\tau h_\tau(t) * \sum_k w_k f_{\Delta D}[D_q(t) - x(t - t_k)]}, \quad (4.9)$$

$$\frac{dD_q}{dt} \approx \frac{q - \sum_k w_k \mathcal{F}_{\Delta D}[D_q(t - t_k) - x(t - t_k)]}{\tau \sum_k w_k \{h_\tau(t - t_k) * f_{\Delta D}[D_q(t - t_k) - x(t - t_k)]\}}, \quad (4.10)$$

both of which can be solved by analog feedback circuits.

Figure 3 illustrates the performance of an analog quantile filter given by equation (4.9) by comparing its quartile outputs ( $q = \frac{1}{4}, \frac{1}{2}$  and  $\frac{3}{4}$ ) with the respective outputs of a digital order-statistic filter in a rectangular moving window  $w_T$  of width  $T$ . In this example, a particular form of a discriminator function

$$\mathcal{F}_{\Delta D}(D) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{D}{\Delta D}\right)$$

is used, and the width parameters  $\Delta D$  and  $\tau$  are chosen to be relatively large in order to better reveal the behaviour of the analog filter.

An important special case of (4.9) and (4.10) is an *analog rank selector*, given by the following equation:

$$\frac{dD_q}{dt} = \frac{Nq - \sum_{k=1}^N \mathcal{F}_{\Delta D}[D_q(t) - x_k(t)]}{\tau h_\tau(t) * \sum_{k=1}^N f_{\Delta D}[D_q(t) - x_k(t)]}. \quad (4.11)$$

The output  $D_q(t)$  of this selector is the  $q$ th rank of  $N$  (independent) signals  $x_k(t)$ ,  $k = 1, \dots, N$ . For example, for  $q = (2N)^{-1}$  the selector will output the *minimum*

<sup>†</sup> Within the framework of a realistic measurement process,  $\tau$  should not be much larger than the characteristic response time of the acquisition system employed for measuring the signal  $x(t)$ .

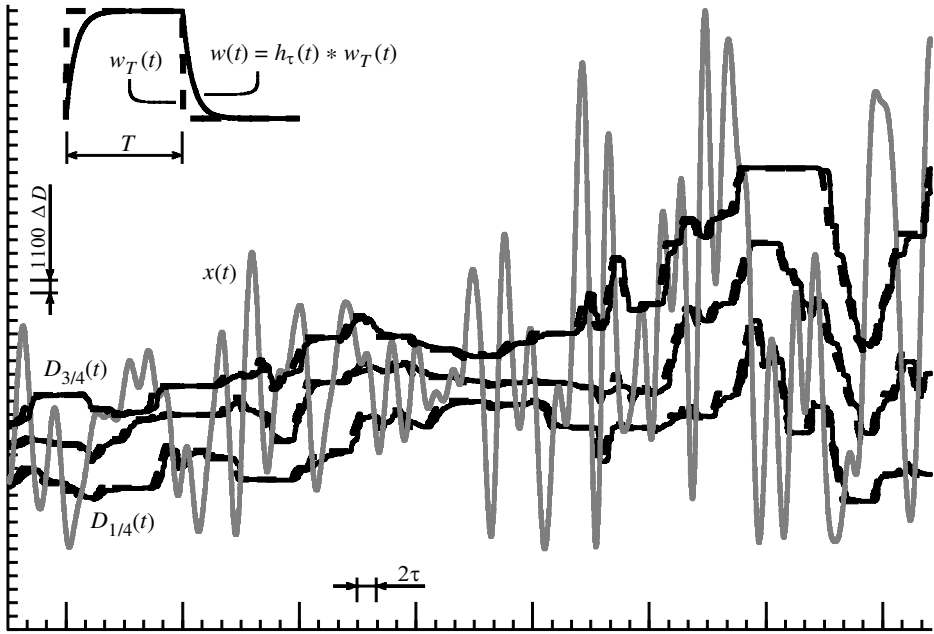


Figure 3. Illustration of the performance of an analog quantile filter given by equation (4.9) by comparing its quartile outputs ( $q = \frac{1}{4}, \frac{1}{2}$  and  $\frac{3}{4}$ , solid black lines) with the respective outputs of a digital order-statistic filter in a rectangular moving window  $w_T$  of width  $T$  (dashed lines).

and, for  $q = 1 - (2N)^{-1}$ , the *maximum* among the incoming signals  $x_k$ . Figure 4 provides an illustration of the performance of an analog rank selector according to equation (4.11) for four signals.

### 5. Counting density and distribution of local extrema

Earlier we showed that the introduction of the threshold distribution of the signal  $x(t)$  and the consideration of a realistic acquisition system enables the development of an analog filter capable of extracting the order-statistic properties of a continuous signal. Note, however, that until now we have only considered the threshold distribution of the signal *amplitude*, which is just one of many characteristics of the signal. Here we show that threshold distributions of other signal characteristics can be introduced in a similar fashion, greatly expanding the scope of meaningful information that can be extracted from the signal.

As an example, let us consider the threshold crossing (also known as zero crossing), or *counting* rate. This characteristic is often used as an indicator of the signal variability and its bandwidth (see, for example, Huang *et al.* 1998; Rice 1944, 1945), and it provides the basis for pulse-height analysis in the acquisition of nuclear radiation spectra (see, for example, Freundlich *et al.* 1947). It is defined as the number of crossings of a given threshold by the signal per unit of time.

Let us consider a single scalar continuous-time signal  $x(t)$ . First, we notice that the total number of counts,  $\mathcal{N}(D)$ , i.e. the total number of crossings of the threshold  $D$  by the signal  $x(t)$  in the time interval  $0 \leq t \leq T$ , can be written as (see, for example,

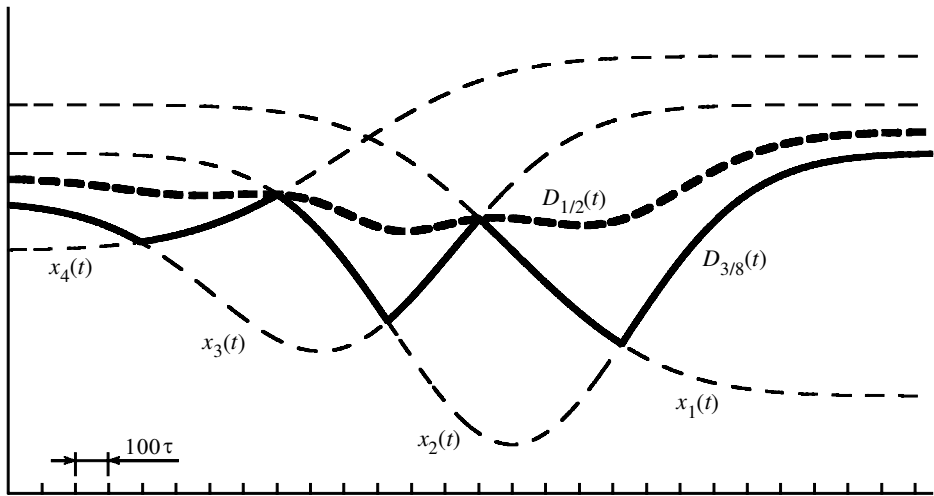


Figure 4. Example of performance of an analog rank selector according to equation (4.11) for four signals ( $x_1(t)$  through  $x_4(t)$ , thin dashed lines). The thick dashed line shows the median ( $q = \frac{1}{2}$  in (4.11)), and the solid line shows the 3rd octile ( $q = \frac{3}{8}$  in (4.11)).

Nikitin *et al.* 1998)

$$\mathcal{N}(D) = \sum_i \int_0^T dt \delta(t - t_i), \quad (5.1)$$

where  $\delta(x)$  is the Dirac  $\delta$ -function, and  $t_i$  are such that  $x(t_i) = D$  for all  $i$ . Using the identity (3.7), we can rewrite equation (5.1) as

$$\mathcal{N}(D) = \int_0^T dt |\dot{x}(t)| \delta[D - x(t)], \quad (5.2)$$

where the dot over  $x$  denotes the time derivative. Thus the expression

$$\mathcal{R}(D) = \frac{1}{T} \int_0^T dt |\dot{x}(t)| \delta[D - x(t)] \quad (5.3)$$

defines the *counting rate* or *threshold crossing rate*.

Replacing the rectangular weighting function in equation (5.3) by an arbitrary moving window  $w$ , the rate of crossing of the threshold  $D$  by the signal  $x(t)$  can be written as the convolution integral

$$\mathcal{R}(D, t) = \int_{-\infty}^{\infty} ds w(t - s) |\dot{x}(s)| \delta[D - x(s)] = w(t) * \{|\dot{x}(t)| \delta[D - x(t)]\}. \quad (5.4)$$

Notice that now the threshold crossing rate depends on time explicitly, and that the rates of upward (+) and downward (−) crossings can be obtained separately by employing the factor  $\theta(\pm \dot{x})$  on the right-hand side of equation (5.4).<sup>†</sup> Note

<sup>†</sup> It is instructive to note that equation (2.7) for the quantile filter can be written in terms of the upward and downward crossing rates as  $dD_q/dt = [\mathcal{R}_+(D_q, t) - \mathcal{R}_-(D_q, t)]/\phi(D_q, t)$ . This provides a basis for yet another implementation of such a filter in an analog device.

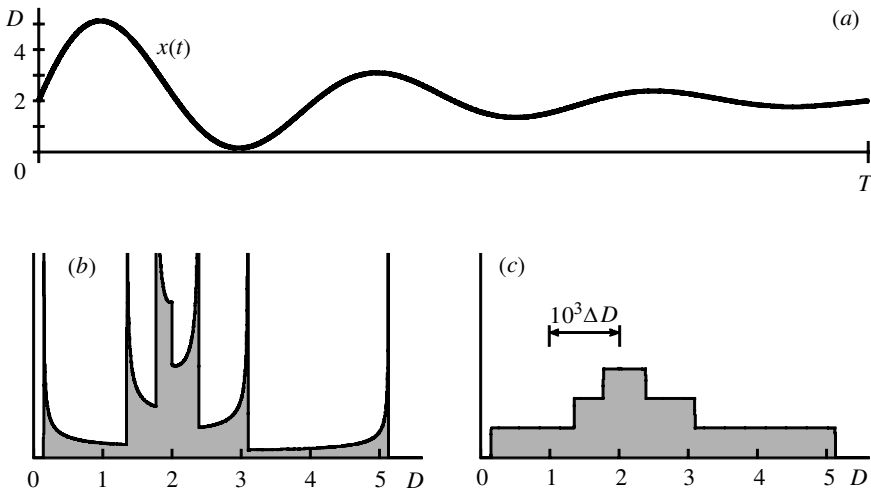


Figure 5. (b) Amplitude and (c) counting densities computed for the fragment of a signal from a damped oscillator, shown in (a).

also that the physical meaning of equation (5.4) is measuring the counting rate by an ideal probe  $f_{\Delta D}(D - x) = \delta(D - x)$  with impulse time response  $w(t)$ . Since  $\int_{-\infty}^{\infty} dD \mathcal{R}(D, t) = w(t) * |\dot{x}(t)|$ , the counting (threshold crossing) density can be written as<sup>†</sup>

$$\rho(D, t) = \frac{w(t) * \{|\dot{x}(t)|\delta[D - x(t)]\}}{w(t) * |\dot{x}(t)|}. \quad (5.5)$$

The significance of the definition of the time-dependent counting (threshold crossing) density, equation (5.5), arises from the fact that it characterizes the *rate of change* in the analysed signal, which is one of the most important characteristics of a dynamic system. Note that, while the amplitude density given by equation (2.11) is proportional to the time the signal spends in the vicinity of a certain threshold, the counting density is proportional to the number of ‘visits’ to this vicinity by the signal. It is also important to notice that, unlike the amplitude density given by equation (2.11), the counting density is not singular,<sup>‡</sup> which significantly simplifies the practical measurements of the counting density. Indeed, it is easy to show that the numerator of equation (5.5) is simply equal to  $\sum_i w(t - t_i)$ , where the sum is taken over all  $t_i$  such that  $x(t_i) = D$ . Figure 5 shows both the amplitude and counting densities computed for the fragment of a signal from a damped oscillator. Note that the amplitude density has a sharp peak at every signal extremum, while the counting density has a much more regular shape.

Employing real instead of ideal discriminators, and the time window  $w = h_{\tau} * w_T$ , we can write an equation for a *quantile counting filter* as

$$\frac{dD_q}{dt} = \frac{w_T(t) * \{|\dot{x}(t)|(q - \mathcal{F}_{\Delta D}[D_q - x(t)])\}}{\tau w(t) * \{|\dot{x}(t)|f_{\Delta D}[D_q - x(t)]\}}. \quad (5.6)$$

<sup>†</sup> See an alternative derivation of equation (5.5), clarifying its physical meaning, in Appendix A.

<sup>‡</sup> Unless, of course,  $w(t) * |\dot{x}(t)| \equiv 0$ .

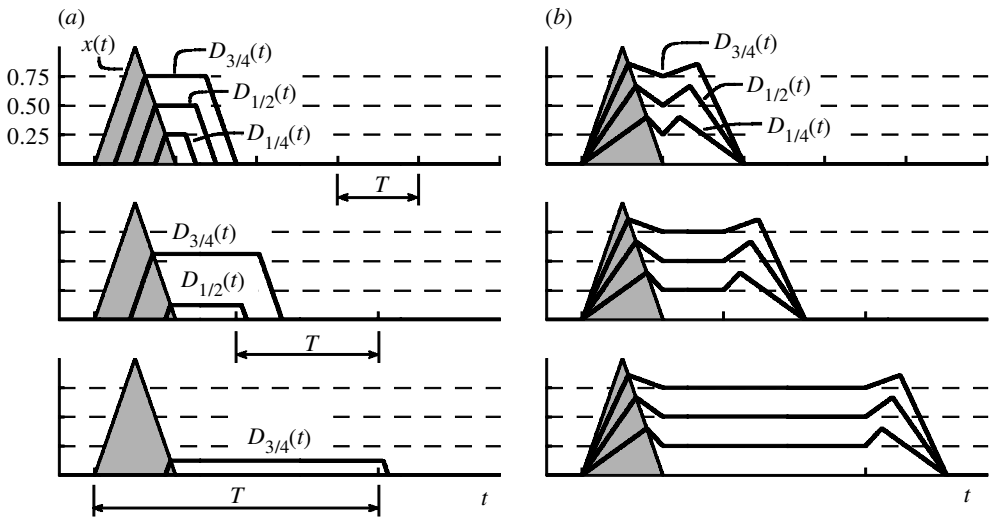


Figure 6. Responses of (a) amplitude and (b) counting quantile filters to a triangular pulse.

Let us illustrate the difference between the outputs of amplitude and counting quantile filters with a geometrical example. Consider, for simplicity, a rectangular time window of width  $T$ . Let us draw the signal  $x(t)$  as a curve in the plane  $(t, D)$ . Imagine that a point travels along this curve from  $t = 0$  to  $t = T$ . The total distance travelled by this point in the horizontal direction will be  $T$ , and the total distance travelled in the vertical direction will be  $\int_0^T dt |\dot{x}(t)|$ . Now, if we draw the horizontal line  $D = D_q$  so that the distance travelled by the point in the *horizontal* direction while staying *below*  $D_q$  is  $qT$ , then  $D_q$  will be the output of the *amplitude quantile filter*. If we draw this line so that the distance travelled by the point in the *vertical* direction while staying below  $D_q$  is  $q \int_0^T dt |\dot{x}(t)|$  (that is, the  $q$ th fraction of the total), then  $D_q$  will be the output of the *counting quantile filter*. Thus the results of amplitude and counting quantile filtering will be fundamentally different. Figure 6 illustrates these differences by showing the responses of the amplitude and counting quantile filters to a triangular pulse. Note that the amplitude filter has a non-zero output only if the duration of the pulse exceeds a  $(1 - q)$ th fraction of the width of the moving window, while the counting filter works in a ‘sample-and-hold’ fashion for the total duration of this window.

#### (a) Counting density for vector signals and modulated threshold density

It is instructive to note that the definitions for threshold crossing rates and densities can be generalized for multi-component (vector) signals. For instance, such a signal can be a two-dimensional vector composed of two scalar signals, which can be of a different physical nature,  $\mathbf{x} = (x_1, x_2)$ . Then, for example, we can measure the rate of crossings of the signal with the threshold  $\mathbf{D} = (D_1, D_2)$ . The expression for such a rate can be written as

$$\mathcal{R}(\mathbf{D}, t) = w * (\sqrt{(\dot{x}_1 \Delta D_2)^2 + (\dot{x}_2 \Delta D_1)^2} \times f_{\Delta D} \{ \sqrt{[(D_1 - x_1) \Delta D_2]^2 + [(D_2 - x_2) \Delta D_1]^2} \}), \quad (5.7)$$

where the width parameter of the probe is  $\Delta D = \Delta D_1 \Delta D_2$ . Note that equation (5.7) describes the rate of crossings of the *vicinity* of the point  $\mathbf{D}$  rather than the rate of crossings of this point itself. This distinction is important for a physical interpretation of the crossing rates of vector signals, since it is meaningless to define the rate of hitting of an infinitesimally small (point) target in more than one dimension.

Such an extension to densities of multi-component signals also allows us to introduce a variety of *conditional* distributions and densities. For example, the choice  $\mathbf{x} = (x, \dot{x})$  and  $\mathbf{D} = (D, 0)$  leads to

$$\phi(D, t) = \frac{w(t) * \{|\ddot{x}(t)|f_{\Delta D_{\dot{x}}}[\dot{x}(t)]f_{\Delta D_x}[D - x(t)]\}}{w(t) * \{|\ddot{x}(t)|f_{\Delta D_{\dot{x}}}[\dot{x}(t)]\}}, \quad (5.8)$$

an expression for the *density of stationary points*<sup>†</sup> in the signal  $x(t)$ , where we have assumed that  $f_{\Delta D_{\dot{x}}}$  is an even function. In practice, a device based on equation (5.8) would constitute an analog implementation of a pulse-height analyser (Nikitin *et al.* 2003).

To conclude this section, let us point out that various threshold densities given by equations (3.2), (5.5) and (5.8) can be viewed as different appearances of a general *modulated threshold density*

$$\phi(D, t) = \frac{w(t) * \{K(t)f_{\Delta D}[D - x(t)]\}}{w(t) * K(t)}, \quad (5.9)$$

where  $K(t)$  is a unipolar modulating signal. Various choices of the modulating signal allow us to introduce different types of threshold densities and impose different conditions on these densities. For example, the simple amplitude density of (3.2) is given by the choice  $K(t) = \text{const.}$ , and setting  $K(t)$  equal to  $|\dot{x}(t)|$  leads to the counting density of equation (5.5).

An expression for the quantile filter for a modulated density can be written as

$$\frac{dD_q}{dt} = \frac{w_T(t) * \{K(t)(q - \mathcal{F}_{\Delta D}[D_q - x(t)])\}}{\tau w(t) * \{K(t)f_{\Delta D}[D_q - x(t)]\}}, \quad (5.10)$$

and the physical interpretation of such a filter depends on the nature of the modulating signal. For example, a median filter in a rectangular moving window for  $K(t) = |\dot{x}(t)|f_{\Delta D_{\dot{x}}}[\dot{x}(t)]$  yields  $D_{1/2}(t)$  such that half of the extrema of the signal  $x(t)$  in the window are below this threshold.

## 6. Threshold densities of spatially dependent signals

Although most of the examples in the previous sections employ simple scalar signals, the consideration of real measurements allows us to treat various other types of signals on a similarly general and consistent basis. In other words, by employing threshold coordinates, we generally can describe a (vector field) variable  $\mathbf{x}(\mathbf{a}, t)$  in terms of its density function in the threshold space, i.e. in terms of a continuous positive-valued function  $\varphi(\mathbf{x}; \mathbf{a}, t)$  such that

$$\int_{-\infty}^{\infty} d^n r \varphi(\mathbf{r}; \mathbf{a}, t) = 1, \quad (6.1)$$

<sup>†</sup> Points  $(t_i, x(t_i))$  are stationary when  $\dot{x}(t_i) = 0$ .



where  $\mathbf{x}$  is the threshold coordinate (an  $n$ -dimensional vector  $\mathbf{x} = (x_1, \dots, x_n)$ ),  $\mathbf{a}$  is a spatial coordinate,  $t$  is time, and  $\int_{-\infty}^{\infty} d^n r \dots = \int_{-\infty}^{\infty} dr_1 \dots \int_{-\infty}^{\infty} dr_n \dots$  denotes the integral over all threshold space. There are numerous ways to construct a continuous density function for a given variable (see Nikitin & Davidchack 2003). The description of a variable in terms of its density function allows us to reformulate many problems of traditional signal analysis and cast them in a form which is readily addressed by methods of differential calculus rather than by algebraic or logical means. This allows a more efficient computational implementation of various signal-processing techniques as well as the implementation of these techniques by continuous action machines.

Consider, for example, a two-dimensional scalar field variable  $x = x(\mathbf{a}, t)$  representing a monochrome image, where  $x$  is the colour intensity at  $\mathbf{a} = (a_1, a_2)$  and time  $t$ . Let us design a filter for removing impulse noise from this image.<sup>†</sup> Assume that we measure the colour intensity by a continuous probe  $f_{\Delta D}$  with the colour resolution  $\Delta D$  and impulse time response  $h_{\tau}(t)$ , and that the effect of the finite spatial resolution of the instrument amounts to spatial averaging with the test function  $f_R(\mathbf{a})$  (spatial impulse response). Then we can write a threshold density function for  $x = x(\mathbf{a}, t)$  as

$$\varphi(D; \mathbf{a}, t) = h_{\tau}(t) * f_R(\mathbf{a}) * f_{\Delta D}[D - x(\mathbf{a}, t)]. \quad (6.2)$$

Thus an equation for a quantile filter can be written as

$$\frac{dD_q(\mathbf{a}, t)}{dt} = \frac{q - f_R(\mathbf{a}) * \mathcal{F}_{\Delta D}[D_q - x(\mathbf{a}, t)]}{\tau h_{\tau}(t) * f_R(\mathbf{a}) * f_{\Delta D}[D_q - x(\mathbf{a}, t)]}. \quad (6.3)$$

Such a filter is highly efficient for removing the dynamic impulse noise as well as static impulse noise from an image, as illustrated in figure 7. In this example, a median filter ( $q = \frac{1}{2}$ ) according to equation (6.3) is used.<sup>‡</sup> Figure 7a shows the original (uncorrupted) image; figure 7b shows the snapshots, at different times, of the noisy image and the respective outputs of the filter. In this example, approximately four out of five pixels in the original image are affected by a bipolar non-Gaussian random noise at any given time. Figure 7c provides an example of removing the static noise (one-third of the pixels of the original image are affected). This example also illustrates the fact that the characteristic convergence time of the filter given by (6.3) is only a small fraction of the time constant  $\tau$ . This is a consequence of the fact that the speed of convergence  $\nu = \{h_{\tau}(t) * f_R(\mathbf{a}) * f_{\Delta D}[D_q - x(\mathbf{a}, t)]\}^{-1}$  is inversely proportional to the density function.

## 7. Discussion

The key component of the approach to the analysis of signals presented here is the introduction of *threshold filters*, where the basic component of a threshold filter is a *discriminator*. Threshold filters allow us to introduce a new independent variable, the threshold variable, and thus provide an additional dimension for the description

<sup>†</sup> Some properties of analog rank filters with respect to additive and multiplicative noise, based on the concept of noise width, are discussed in Ferreira (2001).

<sup>‡</sup> In general, the quantile order of the filter should be chosen as  $q = \Phi_n(0)$ , where  $\Phi_n$  is the amplitude distribution of the noise (either measured or known *a priori*). In the example in this section,  $\Phi_n(0) = \frac{1}{2}$ .

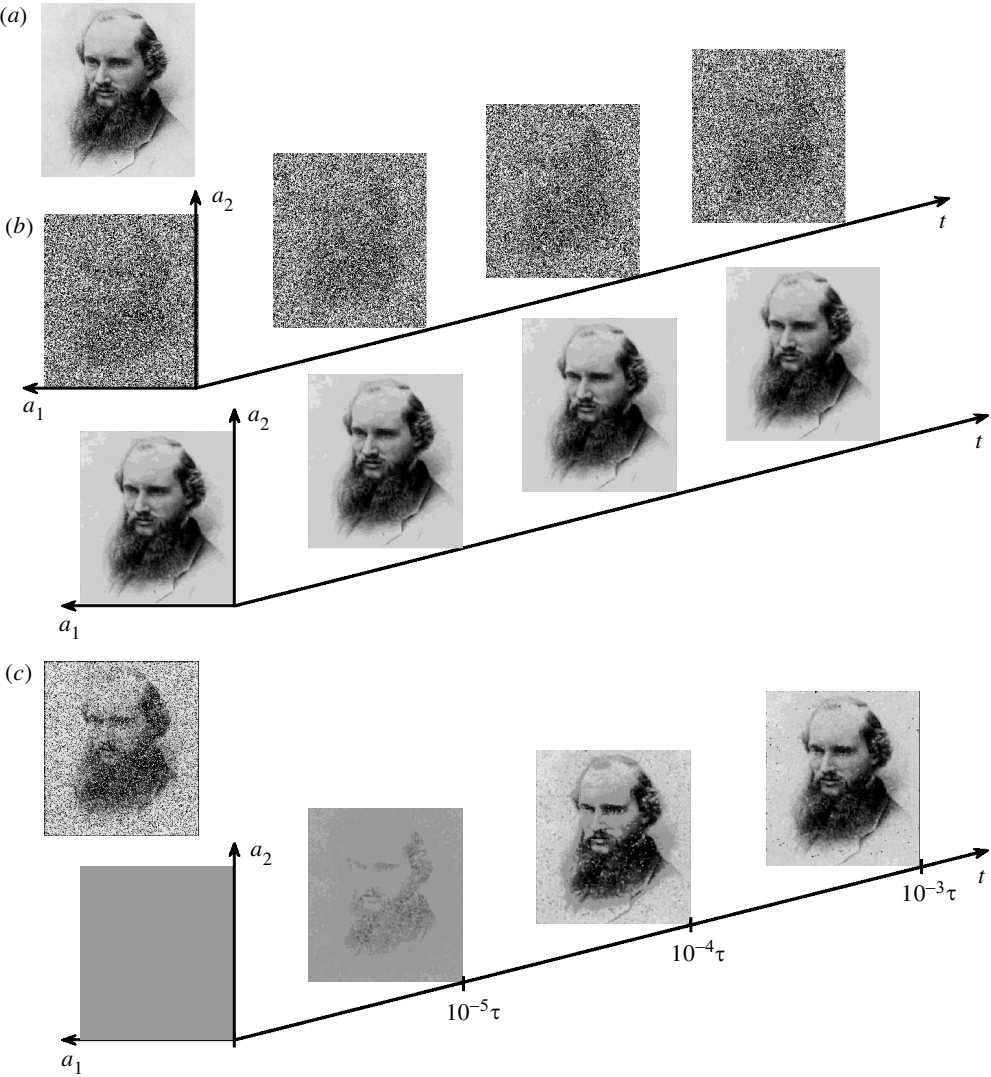


Figure 7. Removing static and dynamic impulse noise from a monochrome image.

of the signal. In addition, by employing discriminators described by a continuous function, we enable differentiation with respect to the threshold variable. By using threshold filters in combination with linear time filters, we represent the analysed signal in terms of the (time-dependent) scalar fields of the threshold variable, and we enable simple implementation of the operations of differential calculus on these fields. Such a representation of a signal in terms of the threshold distributions continuous in *both* threshold and time provides a means for fruitful reformulation of numerous signal-processing tasks. For example, by selecting proper differential operators in combination with threshold and time filters, we can define analog equivalents of digital stack filters, such as those described in Wendt *et al.* (1986). The  $L$  filters and  $\alpha$ -trimmed mean filters of § 3*b* describe two special cases of such analog stack filters.

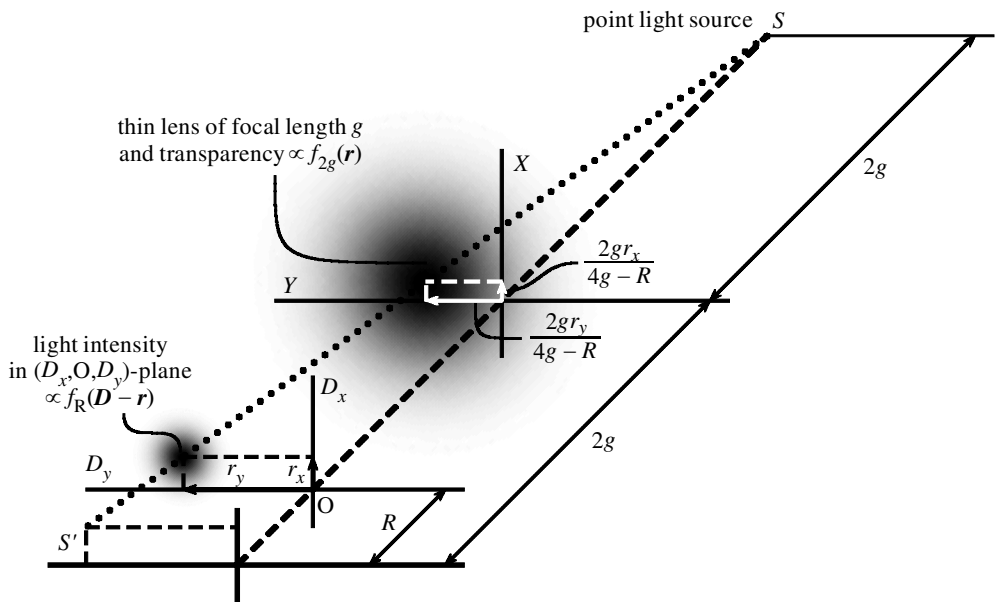


Figure 8. Threshold smoothing filter (probe) for a two-dimensional vector signal.

In this article, we have avoided the discussion of practical hardware realization of threshold filters. Such realization should be quite straightforward for scalar signals. For example, for electrical signals, a threshold filter can simply be a nonlinear amplifier with the input–output characteristic described by a linear combination of various discriminators, the latter being different transconductance amplifiers described, for instance, in Mead (1989) and Urahama & Nagao (1995). It might be more difficult to envision a physical system corresponding to a threshold filter for vector signals. Several examples of such filters are given in Nikitin & Davidchack (2003), and in Appendix B we provide a simplified illustration of a threshold smoothing filter for a two-dimensional vector signal.

Although any signal  $x(t)$ , deterministic as well as stochastic, can be transformed into a threshold distribution function, the formal similarity of the latter with a *probability function* allows us to explore probabilistic analogies and interpretations. For example, we can provide the following probabilistic interpretation of the threshold distribution given by equation (2.3): if  $s$  is a random variable with the density function  $w(t - s)$ , then  $\Phi(D, t)$  is the probability that  $x(s)$  does not exceed  $D$ . Such a probabilistic interpretation allows us to use quantile filters of different orders to construct a variety of (time-dependent) ‘statistical’ estimators of signal properties, like those based on rank tests or linear combinations of order statistics. For instance, *Tukey’s trimean* (Tukey 1977) filter can be defined as

$$D_{Tt}(t) = \frac{1}{2}D_{1/2}(t) + \frac{1}{4}[D_{1/4}(t) + D_{3/4}(t)], \quad (7.1)$$

where  $D_{Tt}$  is the trimean value, and  $D_{1/4}$  and  $D_{3/4}$  are the first and third *quartiles* of the distribution, respectively.

The probabilistic interpretation of the density functions employed in our approach becomes even more fruitful when we consider multivariate density functions. Such an interpretation enables us to construct a variety of ‘rank-test’ estimators of the simi-

larity between a pair of variables in a flexible way, allowing a meaningful adaptation to particular problems (see Nikitin & Davidchack 2003). For example, the quantile function

$$Q(\mathbf{x}; \mathbf{a}, t) = \int_{-\infty}^{\infty} d^n r \varphi(\mathbf{r}; \mathbf{a}, t) \theta[\varphi(\mathbf{x}; \mathbf{a}, t) - \varphi(\mathbf{r}; \mathbf{a}, t)] \quad (7.2)$$

can be given the following probabilistic interpretation. If  $\mathbf{r}$  is a random variable with density function  $\varphi(\mathbf{r}; \mathbf{a}, t)$ , then, for a given  $\mathbf{x}$ ,  $Q(\mathbf{x}; \mathbf{a}, t)$  is the probability that  $\varphi(\mathbf{x}; \mathbf{a}, t)$  exceeds  $\varphi(\mathbf{r}; \mathbf{a}, t)$ . This function can be a highly efficient tool in pattern recognition (A. V. Nikitin, D. V. Popel, R. L. Davidchack and S. N. Yanushkevich 2002, unpublished research).

It is important to realize that even though order-statistic filtering is commonly considered to belong to the domain of signal processing, many other technical fields require similar analyses. For example, consider a solution to the well-known problem of stabilizing the intensity of a light source such as a mercury-vapour lamp or a xenon arc lamp, by controlling only one of the parameters affecting, in a monotonic fashion, this intensity. One can control, for example, only the current. The problem of controlling the intensity of such a source can be described as  $\Phi[I_q(t), t] = q$ , where  $q$  is the desired intensity,  $I_q(t)$  is the (controlled) current, and the explicit time dependence of  $\Phi$  represents the combined effect of other factors such as the voltage and the ambient temperature (A. V. Nikitin 1986, unpublished research). Since the dependence of the intensity on the current is monotonic, one can immediately recognize that this problem is equivalent to finding a  $q$ th rank of the signal  $I(t)$ .

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## Appendix A.

The meaning of equation (5.5) can be clarified by its derivation from another reasoning as follows. Note that a threshold crossing occurs whenever the signal  $x(t)$  has the value  $D$ , and its first time derivative has a non-zero value,  $\dot{x} \neq 0$ . The density of such events can then be expressed in terms of the *joint* density  $\varphi(D, D_{\dot{x}}, t)$  of the amplitudes of the signal and its time derivative as

$$\rho(D, t) = \frac{\int_{-\infty}^{\infty} dD_{\dot{x}} |D_{\dot{x}}| \varphi(D, D_{\dot{x}}, t)}{\int_{-\infty}^{\infty} dD \int_{-\infty}^{\infty} dD_{\dot{x}} |D_{\dot{x}}| \varphi(D, D_{\dot{x}}, t)}. \quad (\text{A } 1)$$

Indeed, the numerator on the right-hand side of this equation is the average absolute value of  $\dot{x}(t)$  at the threshold  $D$ , per infinitesimally small threshold interval  $\Delta D$ . Thus it is the *counting rate*. The denominator on the right-hand side of (A 1) is just the average  $|\dot{x}(t)|$ . The ratio of these quantities is the counting density.

As follows from (2.11), the expression for the joint density is

$$\varphi(D, D_{\dot{x}}, t) = w(t) * \delta[D - x(t)] * \delta[D_{\dot{x}} - \dot{x}(t)]. \quad (\text{A } 2)$$

Substitution of equation (A 2) into (A 1) immediately leads to the expression (5.5) for the counting density.

## Appendix B.

Consider the following simplified illustration of a threshold smoothing filter (probe) transforming a two-dimensional vector signal  $\mathbf{r} = (r_x, r_y)$  into an instantaneous threshold density  $f_R(\mathbf{D} - \mathbf{r})$ , where  $R$  is the width parameter of the probe.

The probe shown in figure 8 consists of a point light source  $S$  and a thin lens with focal length  $g$ . The transparency of the lens is described by  $f_{2g}(\mathbf{r})$ , and the lens is placed in an  $(X, O, Y)$ -plane at a distance  $2g$  from the source  $S$ . Assume that the centre of the lens is at  $2g\mathbf{r}/(4g - R)$ , and we measure the intensity of the light at the location  $\mathbf{D} = (D_x, D_y)$  in the  $(D_x, O, D_y)$ -plane. The latter is parallel to the  $(X, O, Y)$ -plane and is located at a distance  $R$  from the image  $S'$  of the source  $S$ , toward the source. This intensity can be described by  $f_R(\mathbf{D} - \mathbf{r})$ , and thus can be considered an instantaneous threshold density of  $\mathbf{r}$ . Notice that, while the input signal is the *position* of the centre of the lens (times  $(4g - R)/2g$ ) in the  $(X, O, Y)$ -plane, the threshold-filtered signal is the *light intensity* at  $\mathbf{D}$  in the  $(D_x, O, D_y)$ -plane.

If we now cover the  $(D_x, O, D_y)$ -plane with luminophor with the afterglow half-time  $\tau \ln 2$ , then the intensity of light emitted by the luminophor at  $\mathbf{D}$  will correspond to the threshold density  $\varphi(\mathbf{D}, t) = h_\tau(t) * f_R[\mathbf{D} - \mathbf{r}(t)]$ . If  $f_R$  is described by a two-dimensional Gaussian function, then modulation of the intensity of the source  $S$  by  $|\dot{\mathbf{r}}|$  leads to the emitted intensity at  $\mathbf{D}$  being proportional to the rate of crossings of the signal  $\mathbf{r}$  with the threshold  $\mathbf{D}$ , as described by equation (5.7).

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